Cooperation through imitation and exclusion in networks

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\section*{Abstract}

We study the coevolution of networks and action choices in a Prisoners' Dilemma. Agents in our model learn about both action choices and choices of interaction partners (links) by imitating successful behavior of others. The resulting dynamics yields outcomes where both cooperators and defectors coexist under a wide range of parameters. Two scenarios can arise. Either there is “full separation” of defectors and cooperators, i.e. they are found in two different, disconnected components. Or there is “marginalization” of defectors, i.e. connected networks emerge with a center of cooperators and a periphery of defectors.

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\section{1. Introduction}

\subsection{1.1. Motivation}

One of the biggest challenges for economists, biologists, sociologists and other social scientists has been to explain the emergence of cooperative behavior in social dilemma situations. Many researchers share the view that interaction structure is crucial for the emergence of pro-social behavior.\cite{2} Indeed, interaction structure has been demonstrated to matter across a wide array of fields and types of social behaviors. Examples are studies of cooperation and community enforcement in development economics but also studies of crime and drug use in law and economics.\cite{3} Especially if the interaction structure leads to assortative matching it seems that there are good chances for cooperation or other pro-social behaviors to prevail. The underlying idea is that if cooperators interact with increased probability among themselves they can be more successful as a group than defectors. Note that for this argument to work it is important that the interaction structure is either to some degree endogenous or that there is assortative matching ex ante between cooperators and defectors.\cite{4}

There are many empirical findings which show that there is indeed strong assortative matching in social networks.\cite{5} Some recent empirical findings on social networks have highlighted another aspect of the problem. In these studies it has been...
found that people who behave in a more altruistic or generous way are found on average to be more centrally located in social networks. While the evidence for assortative matching is strong and pervasive, the latter effect has only been studied in a handful of studies to our knowledge.\(^6\)

In most of these studies the direction of the causality is unclear. Do agents become more pro-social because they are in a central position in the network or do they obtain this position because their behavior is more socially minded? Similarly, existing theories of assortative matching and group selection fail to answer the question of why interaction structures which favor cooperation should be more likely to emerge. In fact it is likely that there will be multiple feedback effects between action choices and matching.

Eshel et al. (1998), henceforth ESS, have presented a model of cooperation in a circle network. They show that some cooperation can survive if agents imitate successful actions of their neighbors. Imitation is widely recognized to be one of the most important forms of learning in humans and has been shown to be optimal in environments where information is limited or where there are constraints on agents’ reasoning capacities.\(^7\) The result of ESS, though, does not hold for general networks and breaks down (even in a circle) if agents hold some information beyond their interaction neighbors. (See e.g. Goyal, 2007 or Mengel, 2009). Also, since the network is fixed, their model cannot capture possible feedback effects between interaction structure and action choice.

To capture these possible feedback effects we extend their model to analyze the coevolution of interaction structure and behavior. Our model is highly stylized and is based on a few simple principles. Still, it is rich enough to (i) endogenously produce assortative matching and (ii) to contribute to the explanation for why cooperation emerges and why agents displaying less pro-social behavior are often found at the periphery of social networks.

More precisely, we consider agents playing the 2×2 Prisoners’ Dilemma game with their neighbors in an endogenous network. Agents imitate both actions and ‘linking choices’ of their neighbors. Hence, as in ESS, any action is evaluated according to the average payoffs of other agents choosing it. Similarly any potential interaction partner is evaluated by the payoff of other agents linked to him. This is characteristic of many real-life linking decisions, but still our rule will be highly stylized. As an example consider a situation where an agent is deciding to hire (i.e. link up to) a candidate for working together (playing the PD, where cooperation can e.g. be interpreted as providing effort). Imitation of links then says she should look at the payoffs of others that have hired (linked up to) the candidate previously in order to evaluate her possible benefits of this interaction. Other examples include the choice of a doctor where we rely on the experience of others linked to the same doctor or new friends that we link to because our current friends like being linked to them. More precisely, we propose the following imitation learning rules.

- **Agents imitate actions of their neighbors**, i.e. they choose the action (cooperation or defection) with the highest average payoff in their information neighborhood.
- **Agents imitate ‘linking choices’ of their neighbors.** They search new interaction partners locally using information from their neighbors. They are willing to link with another node if and only if the average payoff of others linked to the node in question is high enough.
- **Agents face a fixed capacity constraint.** In this way, any existing link has an opportunity cost that, if high enough, will lead to its replacement.

Those rules are highly stylized rules, which – while they capture some aspects of human interactions – do not account for many other aspects of real-life interactions. An important aspect of such a model of local search is the amount of information that agents have. Indeed, we distinguish between the radii of interaction and of information of the agents, each given by a different parameter. The interaction radius delimits the set of other agents with whom an agent plays the game. Analogously, the information radius determines the set of agents about which an agent has information. These two sets need not coincide, allowing us to cover a wide range of applications. A large information radius (relative to the interaction radius) can reflect situations where relevant information travels easily through the network. Think for instance on the information about one’s friend’s friends or the gossip in a village about some distant neighbors. Situations where relevant information is hard to obtain are represented by a small information radius. The smaller both radii are, the more relevant is the network for the outcome of the game and the learning process.

Given the imitation learning dynamics, we analyze which states of the system are most likely to emerge in the long run. Our main analytical result shows that polymorphic states, i.e. states where both defectors and cooperators coexist, are stochastically stable under some assumptions on the payoff parameters. The topology of the network in stochastically stable states can be of two different types. The first scenario, that we call “full separation” occurs whenever agents hold some information beyond their interaction radius. In this case defectors and cooperators are found in two disconnected components. One component consists of cooperators only, the other of only defectors.\(^8\) The second scenario, that we call

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\(^6\) Leider et al. (2009), Goeree et al. (2010) and Brañas-Garza et al. (2010) all study altruism experimentally (through a dictator game) after eliciting a real life social network. While Brañas-Garza et al. (2010) find evidence that more centrally located people are more altruistic, Goeree et al. (2010) do not report such correlations.

\(^7\) See e.g. Apesteguía et al. (2007); Pingle and Day (1996) or Offerman and Schotter (2009).

\(^8\) This contrasts with models where the network is fixed (like Eshel et al., 1998 or Mengel, 2009). In these models full defection prevails whenever agents hold some information beyond their interaction radius.

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“marginalization” occurs if agents only interact with and hold information about their first-order neighbors. Then networks in stochastically stable states can display a unique polymorphic component. In such a polymorphic component cooperators are found in the center and defectors in the periphery. The linking dynamics in these cases do not lead to full exclusion of defectors, but marginalizes them by driving them out to the periphery of the network (see Fig. 1).

We then simulate the model to gain insight into the importance of different parameters of the model. Confirming our analytical result, we find that polymorphic states do emerge. The share of cooperators in such states increases with the radius of interaction and decreases with the radius of information. Thus, maybe somewhat counter-intuitively, we find that “less information” helps cooperation. Finally we also find that – consistently with empirical findings on social networks – our networks display high clustering coefficients and short average distances.9

The paper is organized as follows. In Section 1.2 we relate our paper to the existing literature. In Section 2 we describe in detail the model, the learning dynamics and the analytical tools used. In Section 3 we present our main analytical results. In Section 4 we present some simulation results. In Section 5 we discuss some of our assumptions in more detail. Section 6 concludes. An appendix contains the proofs of the propositions as well as a glossary with some technical terms.

1.2. Literature

Eshel et al. (1998) have analyzed imitation of behavior when agents are located on a circle. They found that some cooperation in the Prisoners’ Dilemma can survive evolutionary pressures.10 The intuition is that – as agents can only imitate their interaction neighbors – defectors will end up interacting with defectors and cooperators with other cooperators. This reveals the social benefit of cooperation and prevents that cooperators imitate defection. Mengel (2009) and also Goyal (2007) have shown though that this result is not robust. Firstly it does not hold if agents are allowed to hold some information beyond their interaction neighbors, secondly it does not extend to general networks and thirdly it is sensitive to minor changes in the imitation rule.11 Hence, under the general assumptions we use in this paper, action imitation alone cannot sustain cooperative outcomes (except for very particular cases). We show that if the network is endogenous cooperation will survive under many parameter constellations.

In recent years the coevolution of network structure and action choice in games has received increasing attention. Goyal and Vega-Redondo (2005) as well as Jackson and Watts (2002) study the coevolution of linking and action choices in Coordination Games. Both rely on myopic best responses as learning dynamics. Goyal and Vega-Redondo (2005) assume unilateral linking choice (directed network) and find that for high linking costs the efficient action emerges and for low costs, the risk-dominant action. In Jackson and Watts (2002) linking choice is bilateral (undirected graph) and the results are more ambiguous. Skyrms and Pemantle (2000) investigate the dynamics of imitation in a Stag Hunt game, relying on simulation

9 “High” clustering and “short” average distances in the networks literature refers to the comparison of random networks and regular networks. Regular networks have “high” clustering (compared to what would be expected in a random network) and “high” average distances. Random networks are typically said to have “small” clustering and “short” average distances. See e.g. the textbooks by Goyal (2007) or Vega-Redondo (2007).

10 Nowak et al. (1994) have studied cooperation in local interaction models through simulations.

11 Álós-Ferrer and Weidenholzer (2008) in the context of a coordination game played on a fixed network show that the relative size of interaction and information radius can matter.
techniques. Hojman and Szeidl (2006) also study the coevolution of networks and play when the underlying game is a Coordination game.

To our knowledge the coevolution of interaction structure and behavior in the Prisoners’ Dilemma has not been studied analytically, maybe with the exception of few works in the literature on complex networks where mean-field techniques are used to study contagion of ‘bad’ behaviors. 12

One reason is that if best response dynamics is used all outcomes will involve full defection, as defection is a dominant strategy in this game. A way to obtain a non-trivial situation is to study more bounded rational learning dynamics, such as imitation. There are several simulation works studying cooperation in endogenous networks. None of the existing studies, though, can explain why defectors are often found in the periphery of social networks. Simulation studies include Hanaki et al. (2007), Biely et al. (2007), Corten and Cook (2008), Horvath et al. (2008), Zimmermann et al. (2004), Abramson and Kuperman (2001) or Ebel and Bornholdt (2002). Ule (2008) simulates an interesting model of repeated interaction in which agents are forward-looking to some degree.

There is some empirical evidence which is consistent with our main results. Examples are the study by Brañas-Garza et al. (2010). They conduct a two-stage experiment in which they first elicited the (real life) social network of undergraduate students from the University of Granada (Spain) and then measured their altruistic attitudes through a Dictator game. They found that more centrally located subjects (with higher betweenness centrality) are more altruistic. Fig. 1 in their paper nicely illustrates this effect and is reminiscent of Fig. 1 in our paper. In a totally different context Christakis and Fowler (2008) report a major decrease in the prevalence of smoking and a sort of “marginalization” of smokers: while in 1971, smokers were indistinguishable from nonsmokers in terms of integration in their social networks, three decades later, smokers were at the periphery of social networks. However, there are also other studies which do not find (or at least not report) correlations between centrality and altruism. See e.g. Goeree et al. (2010).

2. The model

2.1. The network

There are \( n \) agents, indexed by \( i \), playing a bilateral Prisoners’ Dilemma game with their neighbors in a network. The network is endogenous, i.e. players decide who to form links with. Denote by \( l_i = (l_{i1}, \ldots, l_{in}) \) the vector of linking decisions of player \( i \), where \( l_{ij} \epsilon (0, 1) \). A link \( ij \) is formed whenever \( l_{ij}l_{ji} = 1 \), i.e. if and only if both players “wish” to have the link. Let it be a convention that \( l_{ij} = 0 \), \( \forall i \neq 1, \ldots, n \). The set of all linking decisions \( L = \{l_{i1}, \ldots, l_{in}\} \) and the set of players (nodes) \( N = \{1, \ldots, n\} \) jointly define the network \( G = (N, L) \). Denote by \( \chi \subseteq N \) a connected component of the network, i.e. a maximal subset of nodes s.t. \( \forall i, j \in \chi \) there is a path joining them. 13 The components \( \chi \subseteq N \) define a partition of the network; no agent can be an element of two different components. Finally denote by \( \chi(i) \) the component that contains agent \( i \) and let \( \rho \in \{1, \ldots, n\} \) be the number of components of a network. In Appendix A we define specialist terms for the reader’s convenience.

2.2. Interaction, information and search radii

For any number \( h \epsilon N_+ \), we denote by \( N_i^h \) the set of agents that are within a radius \( h \) of “geodesic” distance to agent \( i \). 14 The set of first-order neighbors of any agent \( i \) is then denoted by \( N^1_i = \{j \neq i | l_{ij}l_{ji} = 1\} \) with cardinality \( \eta_i \). Note that the relation “\( j \) is an element of \( N^1_i \)” is symmetric, i.e. \( j \in N^1_i \Leftrightarrow i \in N^1_j \).

Interaction radius \( Z \). Interactions are not necessarily restricted to an agent’s first-order neighbors. Denote by \( N^2_i \) the set of agents \( j \) interacts with (i.e. plays the bilateral Prisoner’s Dilemma with) or the “interaction neighborhood” of player \( i \). Here \( Z \) is an exogenous, fixed parameter. In some applications one may find it unnatural to interact with agents one is not directly linked with, in others though it seems natural. The model is general, i.e. it allows for either case.

Information radius \( I \). The interaction set \( N^2_i \) will in general not coincide with the set of agents \( i \) has information about. Denote the latter set – the “information neighborhood” of agent \( i \) – by \( N^I_i \). Again \( I \) is a fixed, exogenous parameter. When we say that \( i \) has information about \( j \) we mean that \( i \) knows \( j \)’s average payoff, degree, action choice and the identity of the other players that \( j \) interacts with. As an illustration consider agents on a line with interaction radius \( Z=1 \) and information radius \( I=2 \).

\[
\begin{align*}
N^I_i &\equiv (i-2)(i-1) - i - (i+1) - (i+2) - (i+3) - \\
N^2_i &\equiv (i-3)(i-2) - (i-1) - (i+1) - (i+2) - (i+3) - \\
N^1_i &\equiv (i-3)(i-2) - (i-1) - (i+1) - \\
N^0_i &\equiv (i-3)(i-2)
\end{align*}
\]

12 Fosco et al. (2010), use a cluster mean-field approach to model contagion of misbehaviors. Incentives in their model do not correspond to a prisoner’s dilemma, though. See also Vega-Redondo (2006) for an interesting study of building social capital in a network. Also Fagiolo (2005) or Arenas et al. (2002) study dynamics on networks.

13 A path between \( i \) and \( j \) is a finite set of links connecting \( i \) and \( j \).

14 Geodesic distance and other terms are defined in the Appendix for the reader not familiar with the networks literature.
Let it be a convention that \( N_i^z \) does not contain the player \( i \) herself while \( N_i \) does—i.e. while players do not interact with themselves they have information about themselves. Both \( N_i \) and \( N_i^z \) vary endogenously with changes in the linking decisions of the agents. For example if an agent makes a new friend (link), she will also learn of her new friend’s friends etc., if the size of the information radius exceeds one. One could also think of a situation where one learns about new neighbors of neighbors only after some delay. Since all our analytical results (Propositions 1 and 2) are concerned with absorbing or stable states, they are robust to such a change. Denote by \( n_i(t) \) \((n_i^z(t))\) the cardinality of the set \( N_i \) \((N_i^z)\) at time \( t \).

Search radius \( I+Z \). Revising their linking choices agents search for new partners within their search radius \( I+Z \). Note that these are all the agents they know of, i.e. the agents they have information about (within radius \( I \)) as well as the interaction partners of these agents (within \( I+Z \)). \( N_i^{I+Z} \) denotes the corresponding set. There is also a small probability to link up with a stranger, i.e. a randomly chosen link (See Section 2.4).

As mentioned before the smaller \( Z \) and \( I \) the more important is the network for the outcome of the game and the learning process. As \( Z \) and \( I \) approach the diameter of the network, that is, the largest distance between any two nodes, we approach a global interaction setting. Finally we would like to emphasize again that while our model allows for general \( Z \) and \( I \), the cases \( Z=1 \) and \( I=1 \) are special cases. In other words the model allows for situations where agents interact only with their direct neighbors and where new information can be gained only if they change their direct links.

### 2.3. The game

Individuals play a one-shot symmetric \( 2 \times 2 \) game with their interaction neighbors. The set of actions is given by \( \{C,D\} \) for all players. For each pair of actions \( z_i,z_j \in \{C,D\} \) the payoff \( \pi_i(z_i,z_j) \) that player \( i \) earns when playing action \( z_i \) against an opponent who plays \( z_j \) is given by the following matrix:

\[
\begin{array}{ccc}
   & C & D \\
C & a & b \\
D & c & d \\
\end{array}
\]

We are interested in the case \( c > a > d > b \geq 0 \), i.e. the case where matrix (1) represents a Prisoner’s Dilemma. We assume that all interactions are beneficial \((b > 0)\); i.e. irrespective of \( Z \), all links are worthwhile. Goyal and Vega-Redondo (2005) or Jackson and Watts (2002) have studied cases where not all links are worthwhile. In our model this is not a particularly interesting case to study, as (if \( b > 0 \)) one would always find cooperators and defectors in different components and if \( d < 0 \) cooperation would obtain trivially at least for agents that have some links. We also assume that \( a > (b+c)/2 \), hence cooperation \((C)\) is efficient. The payoffs at time \( t \) for player \( i \) from playing action \( z_i \) when other agents choose actions \( z_{-i} \) and the network is \( G \) are given by

\[
\Pi^I_i(z_i,z_{-i},G) = \sum_{j \in N_i(t)} \pi_i(z_i,z_j).
\]

When choosing an action through the imitation learning process specified below, agents are interested in the average per interaction payoff an action yields in their information neighborhood. This seems the appropriate measure as we assume that agents are myopic and thus choose actions not foreseeing that they might end up having more links in the future as a consequence of their action choice. (We discuss this in more detail in Section 5). Consequently they are interested in whether an action performs well in a given interaction irrespective of whether players choosing this action have many interaction partners or not. Average payoffs (per interaction) for player \( i \) at time \( t \) are given by

\[
\bar{\Pi}^I_i(z_i,z_{-i},G) = \frac{\Pi^I_i(z_i,z_{-i},G)}{n_i^I(t)}.
\]

Note that given Eq. (2) and the assumption that \( b > 0 \) every agent would like to have as many links as possible. In practice, though, there are a large variety of factors (such as time and resource constraints) that limit the “linking capacity” of agents. We summarize such restrictions through the following simple assumption.

**Assumption 1.** No agent can have more than \( \bar{\eta} \in \{3 \ldots n\} \) links.

We assume that \( \bar{\eta} \geq 3 \) to allow a connected network to form. What happens if \( \bar{\eta} = 2 \)? In this case all connected graphs are circles or lines and given the local nature of the search process, any rewiring of the network will quickly lead to the creation of triangles (thus it is not a very much appealing case). Assumption 1 can be rationalized through some strictly convex cost-functions for maintaining links. In the existing literature mostly constant marginal costs for forming links have been assumed with the consequence that equilibrium graphs were either complete or empty.\(^1\)

\(1\) In Eq. (2) agents get the same payoff from all their interaction partners. One could also imagine a situation where – as in the connections model from Jackson and Wolsink (1996) – payoffs are discounted in proportion to the geodesic distance between the two interaction partners.

\(1\) See Goyal and Vega-Redondo (2005) or Jackson and Watts (2002). Jackson and Watts (2002) also consider a capacity constraint in their model of coevolving network and action choices in a coordination game. Whereas in our model a player that has reached the constraint is simply assumed not to want to form links anymore, he can in our model by severing other links.

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will be more realistic than these, but still quite stylized. Before starting to describe the learning dynamics let us introduce some notation.

**Sample payoffs**: Denote by \( \mathcal{P}^t(N_i^t) = \langle n_i^t(t) \rangle^{-1} \sum_{k \in c \mathcal{N}_i(t)} \mathcal{P}_k^t \) the average per interaction payoff of all agents contained in \( N_i^t \) at time \( t \). Analogously denote by \( \mathcal{P}^t(N_i^t \cap N_j^t) \) the average per interaction payoff of all agents in the set \( N_i^t \cap N_j^t \) and by \( \mathcal{P}^t(N_i^t(z)) \) the average per interaction payoff enjoyed by all agents in \( N_i^t \) that choose action \( z \). Let it be a convention that \( \mathcal{P}^t(N_i^t(z)) = 0 \) if \( \text{card}(j \in N_i^t(t)|z_j = z) = 0 \). Since we have assumed that all payoffs are strictly positive, this convention ensures that if an action is not observed it cannot be imitated. Furthermore denote by \( \Pi_{\text{min}}^t(N_i^t) = \min_{j \in N_i^t} \pi(z_i, z_j) \), the minimum payoff that player \( i \) obtains from any of her first-order neighbors.

### 2.4. Learning dynamics

At each point in time \( t = 1, 2, 3 \ldots \) the state of the system is given by the vectors of actions and linking decisions of all agents \( s(t) = ((z_i^t), n_i^t \}_{i = 1} \). Denote by \( S \) the state space. Agents learn about optimal behavior through imitation. As is typical of payoff-based imitation rules, agents substitute an action or link with another one after observing from others that this action or link yields higher payoffs. For action imitation this means copying actions which yield high average payoff in the information neighborhood. In different neighborhoods different actions will yield the highest average payoffs. Hence implicitly agents do take into account the composition of the neighborhood. For imitation of linking choices this means copying links which yield high payoffs in the information neighborhood. More precisely in each period \( t \) the following happens.

1. \( \alpha \) agents are randomly selected to revise their action choice. Each agent \( i \) compares the average per interaction payoff in her information neighborhood of the two actions. If and only if \( \mathcal{P}^{t-1}(N_i^{t-1}(-z_i^t)) > \mathcal{P}^{t-1}(N_i^{t-1}(z_i^t)) \) she changes her action.\(^{18}\) With small probability \( \varepsilon \) she reverses her choice.\(^{19}\)

2. \( \beta \) links \( ij \) with \( j \in N_i^t \cap N_j^t(t-1) \) are randomly selected for revision. If the link \( ij \) does not exist \( i \) and \( j \) can decide to add it. With probability \( 1 - v \) the following decision rule is used. If \( \eta_t(t-1) < \bar{\Pi} \) agent \( i \) chooses \( l_{ij} = 1 \). If \( \eta_t(t-1) = \bar{\Pi} \) agent \( i \) compares the average payoff of the agents interacting with \( j \) that she knows about, \( \mathcal{P}^{t-1}(N_j^t \cap N_i^t) \), to the payoff she derives from her “worst” link, \( \Pi_{\text{min}}^{t-1}(N_j^t) \). If and only if \( \Pi_{\text{min}}^{t-1}(N_j^t) < \mathcal{P}^{t-1}(N_j^t \cap N_i^t) \), she chooses \( l_{ij} = 1 \). Agent \( j \) goes through the symmetric process. If and only if \( l_{ij} l_{ji} = 1 \) the link \( ij \) is added. In this case (and only if \( \eta_t(t-1) = \bar{\Pi} \) agent \( i \) (j) destroys her “worst” link. With small probably \( v \) a randomly chosen link is added or destroyed. Finally any agent exceeding the linking constraint cuts a randomly chosen link.

3. The game (1) is played and agents receive the payoffs.

One may get the impression that the link revision process leads to very sparse networks since sometimes the creation of one link involves the destruction of two others. Note, though, that this is only the case if both agents involved in the link creation are link constrained. If only one of them is link constrained one link is created and one destroyed and if none of them is link constrained then a link is created while none is destroyed. Hence in an absorbing state most agents will operate at their linking constraint. (See also Proposition 1).

To finish this subsection we want to discuss how \( I \) and \( Z \) affect the two dimensions of the learning dynamics. The larger \( I \) the more information agents have. If \( I = Z \) is large the information about the payoffs of the two actions will be of a more “global” nature as \( N_i^t(z) \) will reflect less the local topology \( I \) faces. Under this condition it is also likely, though, that the two sets \( N_i^t \) and \( N_j^t \cap N_i^t \) coincide i.e. that the information agents have about potential new partners is more precise. If \( I - Z \) is small on the other hand information about action payoffs will strongly reflect the local topology but information about potential new partners will be less precise.

### 2.5. Techniques used in the analysis

The learning process described in Section 2.4 gives rise to a finite Markov chain, for which the standard techniques apply. Denote by \( P^t(s, s') \) the transition probability for a transition from state \( s \) to \( s' \) whenever \( e = v = 0 \) and by \( P^t(s, s') \) the transition probability of the perturbed Markov process with strictly positive trembles \( (e, v) \). We make the following assumption on noise.\(^{20}\)

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\(^{17}\) In Section 5.3 we discuss this in more detail. Note also that each agent \( i \) knows only part of any other agent \( j \)’s interaction neighborhood (namely \( N_i \cap N_j^t \)).

\(^{18}\) The notation \( -z_i \) is used to indicate the action not chosen by \( i \).

\(^{19}\) This is the “imitate the best average” rule often used in the literature (Eshel et al., 1998 or Apesteguía et al., 2007).

\(^{20}\) We partly follow Alós-Ferrer and Weidenholzer (2008) in the description of the techniques used.

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Assumption 2. $\epsilon = \xi \nu$ for some constant $\xi > 0$.\footnote{We assume thus (as e.g. Jackson and Watts, 2002) that $\epsilon$ and $\nu$ tend to zero at the same rate.}

An absorbing set under $P^0$ is a minimal subset of states which, once entered is never left. An absorbing state is a singleton absorbing set, or in other words

Definition 1. State $s$ is absorbing $\iff P^0(s,s) = 1$.

As (given that $\epsilon > 0$) trembles make transitions between any two states possible, the perturbed Markov process is irreducible and hence ergodic, i.e. it has a unique stationary distribution denoted $\mu^\epsilon$. This distribution summarizes both the long-run behavior of the process and the time-average of the sample path independently of the initial conditions.\footnote{See for example the classical textbook by Karlin and Taylor (1975).} The limit invariant distribution $\mu^* = \lim_{\epsilon \rightarrow 0} \mu^\epsilon$ exists and its support $\{ s \in S | \lim_{\epsilon \rightarrow 0} P^\epsilon(s) > 0 \}$ is a union of some absorbing sets of the unperturbed process. The limit invariant distribution singles out a stable prediction of the unperturbed dynamics ($\epsilon = 0$) in the sense that for any $\epsilon > 0$ small enough the play approximates that described by $\mu^*$ in the long run. The states in the support of $\mu^*$ are called stochastically stable states.

Definition 2. State $s$ is stochastically stable $\iff \mu^*(s) > 0$.

Young (1993) has shown that stochastically stable states have the property that they minimize the sum of mutations necessary to induce a transition to $s^r$ from any alternative absorbing state. The intuition behind Young’s result is simple. In the long run the process will spend most of the time in one of its absorbing states. Loosely speaking stochastic stability then measures how easy it is to jump from the basin of attraction of other absorbing states to the basin of attraction of state $s$ by perturbing the process through mutations or trembles.\footnote{Ellison (2000) has shown that the time needed to converge to a stochastically stable state $s$ is bound by $O(e^{-\max_{s,s'} X(s,s')})$ where $\max_{s,s'} X(s,s')$ is the maximum over all states of the smallest number of mutations needed to reach state $s$. The resulting wait time can be quite long, which is a criticism often brought forward to this type of models. Note though that – as in our model both action and link imitation occur on a purely local level – the speed of convergence is independent of the size of the population.}

3. Analysis

We first characterize the set of absorbing states of the unperturbed dynamic process. We then provide a characterization of the set of stochastically stable outcomes if the process is perturbed slightly.

3.1. Absorbing states

Our first proposition (which is about the unperturbed process) has three parts. The first part places restrictions on the topology of the networks that can arise in an absorbing state. Due to our different assumption on linking constraints these restrictions will be weaker than those obtained in previous works on the coevolution of behavior and interaction structure.\footnote{The topology most often observed in this literature is the complete graph. See Goyal and Vega-Redondo (2005) or Jackson and Watts (2002).}

Proposition 1 (Absorbing States).

(i) In any absorbing state $\forall i \in N : \eta_i < \pi \Rightarrow \forall j \in N^0_i \cup Z : \eta_j = \pi$.

(ii) States where graphs display only monomorphic components and where (i) holds are absorbing.

(iii) For every $I$ and $Z$, there exists $Z(l) \geq 1$ and a set of payoff parameters $\Psi(i,Z) \neq 0$ s.t. $\forall \pi \leq Z(l)\text{ polymorphic components arise in absorbing states (or sets)}$ for payoffs contained in $\Psi(i,Z)$. In these components the shortest path between any two cooperators never involves a defector.

Proof. Appendix.

If an agent $i$ is not link constrained either all her potential partners must be so or her search set $N^0_i \cup Z$ must be empty. Essentially condition (i) says that agents will maintain as many links as they can. The only reason why an agent $i$ may end up with less than $\pi$ links is that all the agents in her search set $N^0_i \cup Z$ are link constrained and do not want to link with $i$. This means that the density of the network at any absorbing state will be very closely tied to the linking constraint $\pi$. The higher $\pi$, the more dense will the social network be at an absorbing state. If condition (i) holds it is also quite obvious that states where all agents choose the same action are absorbing, as well as polymorphic states where agents that choose different actions are found in different components of the network (ii). The reason is that if all agents choose the same action no new actions can be imitated and no agents have incentives to switch partners since everyone in the monomorphic component earns the same average payoffs.

Part (iii) of Proposition 1 shows that “truly” polymorphic absorbing states exist, in which cooperators and defectors are in the same component and interact with each other. The fact that the shortest path between any two cooperators cannot involve a defector implies for example that it is not possible that defectors are in the center of a component while cooperators...
are in the periphery. Typically, then in all such components the center will consist of cooperators, while defectors are found at
the periphery or that the component. 25 (But not all components where this is the case are part of an absorbing state.) In such
polymorphic states, thus, defectors are not fully excluded from interactions with cooperators, but instead are marginalized at
the periphery of the component. Alternatively it could be that all cooperators are found on one side of the component and all
defectors on the other side of the component with very few links between the two sides (see e.g. Fig. 3). This would also satisfy
condition (iii). We never observe such a component, though, in our simulations described in Section 4.

The conditions on the payoff parameters ensure that no agent is willing to imitate the other action. There must also exist an
upper bound on the interaction radius $Z$ for which such states can be absorbing. To see this note that as $Z$ increases we
approach a scenario where all agents in the same component interact, i.e. where the network does not matter anymore for
matching. In this case defection, being a dominant strategy, will always yield higher average payoffs.

Fig. 2 illustrates the dynamics towards an absorbing state where defectors are driven to the periphery.

Why can there be no defector on the shortest path between any two cooperators? This is largely a consequence of local
search. First note that any cooperator $i$ linked to a defector $k$ is always willing to substitute this link for a link with one of $k$'s
interaction neighbors (irrespective of the action that neighbour is taking). 26 If such a neighbor $j \in N_1^i$ is herself defecting she
will want to link with $i$, if (except for $i$) she observes only defectors. In this case $P^i_{\min} (N_1^i \cap N_1^j) \geq P^i_{\min} (N_1^j)$. The link $ji$ will be established

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25 We will use the terms center and periphery loosely according to their meaning in common language and not according to their graph-theoretic definition.

26 The payoff she obtains from the defector $P_{\min} (N_1^i) = b < P^i (N_1^1 \cap N_1^j)$ no matter what action $j$ is taking, as $j$ is linked to at least one defector $i$ knows
about (namely her own first-order neighbor).
and the links ik and jk will be severed. Repeating the argument it can be shown that any two cooperators connected through a path of defectors will at some point find each other and form a link. This is an immediate consequence of local search. But then typically defectors will eventually end up in the periphery of the component. Note also that because of the local search process, where agents meet each other explicitly through common neighbors, all graphs will display a high degree of clustering.

To illustrate further how it exactly works that cooperators who are initially not in each others information neighborhood will find each other, let us consider the following mini-network of 10 players (Fig. 3). In the mini-network the white nodes 1, 4, 5, 7 and 10 are cooperators and the remaining players defectors. For the sake of illustration we want to uniquely focus on link-imitation. Hence we are holding the actions of these players constant. In the mini-network each player is constraint to have at most 3 links. We assume that \( I = Z = 1 \). Note also that initially (upper left panel) the defectors are more central, in the sense that the shortest path between any two defectors does never involve a cooperator, while the reverse is not true, i.e. the shortest path between any two cooperators may involve a defector. Now we would like to show how all cooperators do find each other. Assume that initially players 2 and 4 can form the link 24. Both players will have incentives to do so. The cooperator 4 is not linking-constraint, but even if she were she would prefer to cut a link with a defector for the link 24 which promises her a payoff of \( (c + 2d) / 3 \), the same that it promises to player 2 through their mutual acquaintance player 3.\(^{27}\) Hence player 2 is willing to cut a link to say player 6. Similarly, if players 8 and 10 are offered to link they will find it in their interest to do so (left middle panel). If next, players 1 and 4 and players 7 and 10 are, respectively, offered to link, they will find it again convenient to do so and so on.\(^{28}\) The remaining figures show how all cooperators end up linked to each other.\(^{29}\) (In the last picture of Fig. 3 the nodes have been rearranged.) Note that here (while condition (iii) is satisfied) defectors are not “more in the periphery than cooperators”. This is due to the small number of agents in our mininetwork. If one imagines such processes taking place everywhere in a large network one can imagine why defectors will tend to be in the periphery. The networks in Fig. 2 show the more realistic picture of a larger network. The mini-network is intended to make the reader understand how the process of unraveling works and to see why a situation where cooperators are separated from each other through a path of defectors cannot be stable. Note that this strict assortativity arises in spite of the fact that initially the cooperators (1, 5 and 7) and (4 and 10) are not in each other’s search set.

3.2. Stochastically stable states

Proposition 1 delimits the set of states that can potentially be stochastically stable, since (as explained in Section 2.5) every such state must be absorbing for the unperturbed dynamics. We are ultimately interested in the set of stochastically stable states. Our main result is Proposition 2.

**Proposition 2 (Stochastically stable states).** There exists a threshold level \( a^*(Z, I, \eta) \in (d, c) \) s.t.

1. If \( a \geq a^*(Z, I, \eta) \) all stochastically stable states are polymorphic and consist of at most two components.
Appendix). Goyal and Vega-Redondo (2005). It remains to show that the reverse is not possible which involves a bit of a lengthier argument. (See Lemma 1 in the Appendix.

Proof. Appendix.

Stochastically stable states are either polymorphic (meaning that both cooperators and defectors are present) or characterized by full defection. If \( I + Z > 2 \) monomorphic states are connected and polymorphic states consist of two monomorphic components. A sufficient condition for polymorphic states to emerge is that the payoff for joint cooperation is high enough. How high that depends on the number of links \( \pi \) each node can maintain and on the information \( I \) and interaction radii \( Z \). If in addition \( I + Z > 2 \), then all stochastically stable states will consist of two monomorphic components, one of defectors and one of cooperators. In these cases there is full exclusion. If \( I + Z = 2 \) on the other hand, graphs in stochastically stable states can be “truly” polymorphic. Since all stochastically stable states must be absorbing under the unperturbed process the shortest path between any two cooperators can never involve a defector in such polymorphic components.

What is the intuition for this result? The tension in the Prisoners’ Dilemma arises from the fact that while defection is a dominant strategy, cooperation provides the highest benefit to a community (is efficient). This is all the more so the higher the payoff parameter \( d \in (d,c) \). Cooperation then will emerge as a stable outcome of the imitation learning process if cooperators interact with increased probability among themselves, i.e. if there is strong assortativity. This reveals the social benefit of cooperation and induces other agents to imitate cooperators. The most extreme situation is a state where cooperators and defectors coexist in two different components of the network. Two forces in our model facilitate that the process arrives at such a situation. Firstly as action imitation occurs among one’s information neighbors only, defection will spread locally. Secondly as new links are searched locally (at a radius of \( I + Z \)), cooperators can avoid the interaction with defectors in their interaction neighborhood by cutting these links and linking up with other cooperators. If the payoffs for defection are “too high” cooperators will tend to imitate defectors and cooperative components can easily be destabilized. But what does “high” mean exactly? This depends on the relative size of the interaction and information radius \( (I,Z) \) as well as on the number of links \( \pi \) each node can maintain.

The relative size of the information radius \( I \) (relative to \( Z \)) has a double effect on the dynamic process. A smaller information radius \( I \) (relative to \( Z \)) forces defection to spread more “locally”, since anyone imitating a defector has to be within radius \( I \) of that defector. This helps cooperation by making it more likely that defectors interact with each other and obtain the inefficient payoff for joint defection \( d \). On the other hand a higher information radius \( I \) (relative to \( Z \)) improves the information agents have about potential partners inside their search radius \( I + Z \) making it more easy for them to exclude defectors from beneficial interactions with cooperators.

Why do we obtain few (at most two) disconnected components in stochastically stable states? Note that one linking tremble, whereby two agents in different components form a link, suffices to connect any two disconnected components in which agents choose the same actions. Moreover, this linking tremble can be such that after the tremble the unperturbed dynamics leads to the creation of more links (for example by connecting two agents one of who has a neighbour which is not link-constrained). If this is the case, then the transition cannot be reversed by a single tremble. This is the underlying reason why bigger components tend to be more stable.

Next we would like to address the question of why with a larger search radius societies tend to be disconnected while if \( I + Z = 2 \) they are connected? The intuition is as follows. Note that in the process of marginalization of defectors there are always some cooperators which are in a rather bad situation since they are interacting with several defectors (see Fig. 1). On the other hand their role in society is important since they build a bridge between cooperators and defectors. With a larger search radius new agents are able to link up to more distant partners and hence cooperators in such a situation will find it easier to find other partners to link to. This tends to lead to separation of cooperators and defectors. The sharp transition at \( I + Z = 2 \) arises, since in this case even if bridging cooperators (because of an action tremble in the network) link up to a new interaction partner, they will always link up with a neighbor of one of their “bridge contacts”. This process can occur repeatedly and the identity of the bridging cooperator may change during the process. Eventually, though, it will get stuck in a cycle. If \( I + Z > 2 \) on the other hand bridging cooperators (after an action tremble in the network) can keep linking up with new agents until they find another bridging cooperator. Once they have found such a cooperator and linked up with him the two components will split.

Note that there is also some role of the capacity constraint. Capacity constraints reflect convex linking costs. Without capacity constraints (and if \( b > 0 \)) agents will never find it optimal to cut any link and hence the unperturbed dynamics will always lead to the complete network (once the network is connected). With respect to actions, we will find that universal defection is the only outcome in this case. The reason is that the only effective means that cooperators have to create some assortative matching relies on exclusion of defectors. If no links are ever cut, though, then the unperturbed dynamics cannot

\[\text{(ii) if } a > a^*(Z,I,\pi) \text{ and in addition } I + Z > 2 \text{ then all stochastically stable states are polymorphic and consist of exactly two components.} \]
\[\text{(i) if } a < a^*(Z,I,\pi) \text{, then all stochastically stable states are characterized by full defection and consist of one component.} \]

---

30 The effect of parameters \( I \) and \( Z \) will be illustrated further in our simulations in Section 4.

31 For the reader familiar with graph-theoretic techniques a more formal argument goes as follows. Consider a state \( s^* \) and a state \( s \) obtained from \( s^* \) by one linking tremble. Then starting from an \( s^* \)-tree cut the arrow leading away from \( s \) and add the arrow \( s \rightarrow s^* \). This will create an \( s \)-tree where we have added a link at cost one and destroyed a link of cost at least one. Hence there is at least one \( s \)-tree with stochastic potential of at most that of \( s^* \). See e.g. page 190 in Goyal and Vega-Redondo (2005). It remains to show that the reverse is not possible which involves a bit of a lengthier argument. (See Lemma 1 in the Appendix).
create such assortative matching. Hence if a single cooperator turns to defection this will induce defection in the whole network. Of course this need not always be a perfect description of reality. In a world without capacity constraints, the complete network need not always obtain (empirically) and it can sometimes also happen that agents cooperate even though the network is complete. From a theoretical point of view, though, we find it more interesting to study the case with capacity constraints.

A related question is what would happen if agents searched for new links globally. Note that as in this case the sets $N_i^I \cap N_i^J$ can be empty an additional rule is needed to evaluate potential new links. Irrespective of the specific form of such an additional rule, though, the results with global search will change. Several simulations we performed show that the process can be empty an additional rule is needed to evaluate potential new links. Irrespective of the specific form of such an additional rule, though, we find it more interesting to study the case with capacity constraints.

The network is complete. From a theoretical point of view, though, we find it more interesting to study the case with capacity constraints.

We have seen that while fully cooperative states will not be observed polymorphic states can often occur. The condition needed is that the payoff for joint cooperation is high enough, where the last qualification depends on many parameters of the matching scenario. Hence both local search together with capacity constraints are crucial ingredients of the model.

The aim of the next section is to develop a better intuition for the role of our different model parameters.

4. Simulation Results

In this section we illustrate and complement the analytical results through simulations. We explore essentially two aspects. First (under payoff parameters where polymorphic structures are "likely" to emerge) we show the effect of $(\beta/z)$, $I$ and $Z$ on the fraction of cooperators denoted by $\phi_c$. Remember that $z$ denotes the number of agents revising their actions in each period and $\beta$ denotes the number of links revised in each period. Hence $(\beta/z)$ could be called the relative speed at which the network evolves. We address the question separately for $I+Z>2$ (Table 2) and $I+Z=2$ (Table 1). The difference between both cases is that when $I+Z>2$, stochastically stable polymorphic states are always composed of two separate components. If $I+Z=2$, there can be stochastically stable states with polymorphic components, like those illustrated in Fig. 1. Second, we measure the effect of the search radius $(I+Z)$ on the topology of the network, in particular with respect to average clustering and average distance within components.

In all the simulations that we report here there are $n=400$ nodes. The initial network is random and satisfies $\eta_i \leq \eta_i^f$. The initial number of cooperators is 0.5n (randomly placed on the network). Payoff parameters are chosen such that for any $I,Z,(\beta/z)$ polymorphic structures are "very likely" to emerge ($c=1,a=0.9,d=0.1,b=0$). We choose $z=1$ and $\beta \in \{1,5,10\}$. The combinations of $(I,Z)$ analyzed are $(1,1),(1,2),(1,3),(2,1),(3,1))$. Simulations include (small) noises $\epsilon$ and $\nu$. For each case, we perform 100 realizations of the dynamic process, and for each realization we obtain the average of the fractions of cooperators in the last $2 \times 10^3$ time steps. This time-average is then an independent observation of $\phi_{ci}$, thus for each parameter constellation we have a sample of 100 independent observations. The 95% confidence interval for $\phi_{ci}$ is constructed on the basis of these 100 independent observations, where the sample fraction of cooperation is $\hat{\phi}_c = \sum_{i=1}^{100} \phi_{ci}/100$.

**Result 1.** If $I+Z=2$, all realizations produce a network where the largest component consists of a core of cooperators with defectors lying on the periphery. The parameter $\beta$ has almost no effect on the fraction of cooperators (see Table).

<table>
<thead>
<tr>
<th>Table 1</th>
<th>Share of cooperators if $I-Z$.</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\beta$</td>
<td>Interval for $\phi_c$ (95%)</td>
</tr>
<tr>
<td>1</td>
<td>[0.42,0.54]</td>
</tr>
<tr>
<td>5</td>
<td>[0.41,0.53]</td>
</tr>
<tr>
<td>10</td>
<td>[0.43,0.55]</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Table 2</th>
<th>Shares of cooperators if $I&gt;Z$.</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\beta$</td>
<td>Interval for $\phi_c$ (95%)</td>
</tr>
<tr>
<td>1</td>
<td>$I=1; Z=2$</td>
</tr>
<tr>
<td>1</td>
<td>[0.15,0.32]</td>
</tr>
<tr>
<td>5</td>
<td>[0.31,0.50]</td>
</tr>
<tr>
<td>10</td>
<td>[0.40,0.59]</td>
</tr>
</tbody>
</table>

Results are available upon request.

\(t_{\text{max}}\) is the total number of timesteps of each simulation. $t_{eq}$ is the timestep s.t. the system approximately equilibrates. $t_{\text{max}}$ depends on $(I,Z)$. For small noises $\nu=10^{-4}$ and $\epsilon=10^{-4}$, we found by inspection of the time series of $\phi_c$ that $t_{eq}<3 \times 10^3$. We set $t_{\text{max}}=4 \times 10^4$ in all cases that we report here.

In Fig. 2 we have shown a typical example. Intervals are asymptotic, with $\phi_{ci} \in [\phi_{ci} - 1.96, \phi_{ci}(1-\phi_{ci})/100, \phi_{ci} + 1.96, \phi_{ci}(1-\phi_{ci})/100]$. $\phi_{ci}$ is the average over 100 realizations’ fractions of cooperators (i.e. 100 Monte Carlo (MC) simulations) that in turn are averaged over the last $t_{\text{max}}-t_{eq}$ timesteps in each MC simulation.

The intuition for this result is as follows. If $I + Z = 2$ imitation of defection will necessarily lead defectors to interact with each other reducing their average payoff. The action imitation dynamics itself is able to limit the spread of defection, which irrespective of the value of $\beta$ can only invade a small group of agents. The linking dynamics then tends to push these defectors towards the periphery of the network.

**Result 2.** If $I + Z > 2$ the fraction of cooperators increases with $\beta$ and tends to increase with $Z$ and decrease with $I$ (Table 2).

To illustrate this result, we show in Table 2 the intervals for $\varphi_c$ and in Fig. 4 the observed distribution of $\tau_c$ for each sample. Panels (a)–(d) show the effect of $\beta$, while (e) and (f) show the effect of $I$ and $Z$, respectively.

What is the intuition for this result? If $I + Z > 2$ higher values of $\beta$ increase the fraction of cooperators. Since action imitation in these cases allows for the infection of “many” agents with defection, exclusion ($\beta$) is very effective in raising the number of cooperators. Consider first the cases $I = 1$ and $Z > I$ (panels (a), (b), (e)). Cooperation has good chances, as the small information radius forces defectors to interact with each other after action imitation. On the other hand though (as $Z$ (and thus $Z + I$) is “large” relative to $I$) the quality of information about potential new links is relatively bad and the linking dynamics leads to more “erroneous” new links. This is why the effect of $\beta$ is relatively less important in the case $I > Z$ compared to the case where $I < Z$. Now consider the case where $Z = 1$ and $I > Z$ (panels (c–f)). Being informed is not per se good for cooperation. Indeed, since agents imitate average behavior in this radius, the higher is $I$ the more probable it is that a cooperator imitates defection. On the other hand if $I$ is high, the linking dynamics is more accurate due to the higher quality of information and less “erroneous” choices are made. Inspecting overall cooperation rates, it can be seen that the negative effect of $I$ on the action imitation process dominates the positive effect of $I$ on cooperation through the linking dynamics. The latter effect though explains that $\beta$ has a higher “marginal” effect in the cases where $I > Z$ (compared with $I < Z$). Next we want to show some results on topology.

**Table 3**

Average distances and clustering.

<table>
<thead>
<tr>
<th>$I + Z$</th>
<th>$\tau(i)$</th>
<th>$J(i)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>0.531</td>
<td>7.6</td>
</tr>
<tr>
<td>3</td>
<td>0.237</td>
<td>5.8</td>
</tr>
<tr>
<td>4</td>
<td>0.088</td>
<td>4.1</td>
</tr>
</tbody>
</table>

Result 3. Graphs obtained display an average clustering coefficient and average distances that are both decreasing with $I + Z$.\textsuperscript{36}

Result 3 confirms our intuition about some of the topological features of the components created by the dynamics. Average clustering ($\bar{Z}(i)$) and average distance ($\bar{d}(i)$) are both decreasing with the search radius $I + Z$ (see Table 3). The search radius represents the extent of the locality in linking dynamics. When $I + Z$ is low, the probability that two first neighbors of any agent $i$ are connected themselves is very high, but since links are concentrated within a small radius, the average distance between two nodes is large. When $I + Z$ is high, since each agent has more possible partners, the probability of choosing a second neighbor decreases (and so does the average clustering). But on the other hand links with nodes that are relative far away are shortcuts that reduce average distances. Note that these features are independent of $\beta$ and of the particular combination of $I$ and $Z$.

5. Imitation learning—discussion of assumptions

Agents may rely on imitation learning for a variety of reasons. First and foremost it may be a form of boundedly rational behavior. But there are also situations where it may be a good idea to rely on imitation learning. One reason could be to save calculation and decision-taking costs, another to aggregate heterogenous information or to free-ride on the superior information of others. It can also be optimal if information is severely restricted. (See Alós and Schlag, 2009 as well as the experimental evidence discussed above). In our context we view imitation learning mostly as a form of boundedly rational behavior, but it should also be clear that our environment is very complex. Hence in the presence of reasoning costs imitation may not be a bad idea after all. For example it is not clear what the best strategy is in a neighborhood where some agents cooperate and some defect and where the decision maker has no information about what happens outside her information neighborhood. Rational decision making in such a situation would require the agent to form beliefs about the probability of all possible future paths of play given her linking and action choice at $t$ and then choose optimally. These beliefs would have to be updated at each $t$ according to the new information received. This implies huge demands on the rationality of decision makers and imitation may make a lot of sense in such a complex environment.\textsuperscript{37} Imitation basically says that when trying to evaluate which action would be best in a given neighborhood agents look at the payoffs others (and themselves) obtain when choosing different actions. Such imitation is not explicitly conditional on the composition of the neighborhood, but implicitly the ranking of average payoffs will depend on it. A more sophisticated model could be one where agents weigh the information according to others according to the (perceived) degree of overlap of interaction neighborhoods or according to their distance (if $I > 1$). We decided for this more simple process to study the effect of imitation learning per se.\textsuperscript{38}

One consequence of assuming that agents are imitators is that they are not forward looking. This is also the reason why we assume that agents consider average instead of total payoffs when deciding to choose an action. In particular when choosing an action myopic agents take as given the cardinality of their interaction neighborhood and thus should be interested in the average payoff per interaction. Using total payoffs, though, would imply that they anticipate having more (or less) links in the future as a consequence of their action choice. While we think that it is inconsistent with imitation to focus on total payoffs, let us still outline what would happen in this case. It should be clear that the characterization of absorbing states (Proposition 1) will be unaffected. The reason basically is the homogenous linking constraint which in our absorbing states implies that considering total vs average payoffs does not affect the ranking of the two actions. In terms of stochastic stability one has to be a bit more careful. Here it seems that if at all such an assumption would tend to favor cooperative outcomes, since the linking dynamics (during a transition) can favor cooperators who tend to have more links during transitions. We conducted a number of additional simulations focusing on $Z = I = 1$ where we change the action imitation process to take into account (average) total payoffs of all defectors/cooperators rather than average payoffs of defection/cooperation. The results of these simulations show that qualitatively our results are robust, while there is a higher fraction of cooperators if total payoffs are used.\textsuperscript{39} The main argument that speaks against this alternative assumption is that it would be inconsistent with myopic imitation learning as it is commonly used in the Economics literature.\textsuperscript{40}

6. Conclusions

We develop a simple model to study the coevolution of interaction structures and action choices in Prisoners’ Dilemma games. Agents are boundedly rational and choose both actions and interaction partners through payoff-based imitation. We

\textsuperscript{36} We measure these characteristics on the largest component.

\textsuperscript{37} Interestingly Duersch et al. (2010) in a recent paper show that imitation cannot be outperformed by any other decision rule (including much more sophisticated ones) in a large class of two player games. See also Schipper (2009) or Hehenkamp and Kaarbøe (2008).

\textsuperscript{38} Also both Propositions seem robust to these possible extensions. In the case of Proposition 1 this should be easy to see. The intuition for why Proposition 2 should still hold is as follows. Essentially the stochastic potential of the different states is determined by action or link trembles where a single defector/cooperator invades an entirely defective/cooperative neighborhood. But then the payoffs of the single defector/cooperator are unaffected and (because of the structure of the PD) always higher or lower than that of any convex combination of payoffs for the other action.

\textsuperscript{39} Results are available upon request.

\textsuperscript{40} Fernando Vega-Redondo (2003) in his textbook Economics and the Theory of Games describes imitation learning as follows "...reflects the idea that a player can only observe (or assimilate) information on the action chosen and payoff received in the preceding period by other players. Then taking this information as descriptive of what may be expected of the future (i.e. holding static perceptions) the player is supposed to imitate an action...".
find that polymorphic states can emerge for some parameters of the model. Whenever agents hold some information beyond their interaction partners defectors and cooperators will never interact in stochastically stable states, i.e. they are found in disconnected components. Otherwise graphs in stochastically stable states can consist of a core of cooperators with defectors lying on the periphery of the component. Consistently with empirical findings on social networks, the networks we obtain display high clustering coefficients, short average distances, strong assortative matching and explain the empirical finding that agents with less altruistic behaviors tend to be less central in social networks.

Acknowledgements

We wish to thank Vincent Buskens, Sven Fischer, Sanjeev Goyal, Debraj Roy, Javier Rivas, Marco van der Leij, two anonymous referees and especially Fernando Vega-Redondo for very helpful discussions and seminar participants in Belfast, Carnegie Mellon (NASM 2008), Cornell, Jena, Leuven, Maastricht (CTN workshop 2009), Milan (ESEM 2008), Nuernberg, Paris and Utrecht for their comments. Mengel gratefully acknowledges support from the Spanish Ministry of Education and Science (Grant SEJ 2004-02172) and the European Union (Grant PIEF-2009-235973). Fosco acknowledges the support of the Núcleo de la Iniciativa Milenio “Ciencia Regional y Políticas Públicas” (Chile).

Appendix A. Some definitions

In this section we provide definitions for a number of terms in alphabetical order.

Action: An action in our model is to choose either C or D.

Bilateral linking choices: Bilateral linking choice means that consent is necessary to establish a link, i.e. the existence of a link from i to j does imply the existence of a link from j to i.

Average clustering or clustering coefficient: The average clustering in a network is obtained by averaging clustering across all i \in N.

Average distance: The average distance is the average geodesic distance separating pairs of nodes i and j between which a path exists. The average distance is always computed within a connected graph (see e.g. Vega-Redondo, 2007).

Clustering: The clustering of a node i is defined as the fraction of pairs of (first-order) neighbors of i which are themselves (first-order) neighbors.

Component: We denote by \chi \subseteq N a connected component of the network, i.e. a maximal subset of nodes s.t. \forall i,j \in \chi there is a path joining them.

Directed graph: A directed graph results from unilateral linking choice.

Geodesic distance: The geodesic distance between nodes i and j is defined as the minimum number of links that need to be used along some network path to connect i and j. If no such path exists the usual convention is to say that the geodesic distance is infinite.

Linking choices and links: We denote by l_i = (l_{i1}, \ldots, l_{in}) the vector of linking decisions of player i, where l_{ij} \in (0,1) and l_{ij} = 1 means that i desires the link with j. A link ij is formed whenever l_{ij}l_{ji} = 1, i.e. if and only if both players “wish” to have the link.

Monomorphic: We say a network or a component of a network is monomorphic if all players in the network (component) choose the same action.

(Not) link constrained: We say an agent i is link constrained if \eta_i = \eta. We say an agent i is not link constrained if \eta_i < \eta.

Radius: We say that node j is within radius x of node i if the geodesic distance between i and j is at most x.

Realization: A realization of a random variable is the value of the random variable that is actually observed, i.e. a draw from the random variable.

Note that each run of our simulations can be conceived as the realization of a random variable where the random variable specifies the initial condition as well as which nodes are drawn for action revision and which links are drawn for link revision in each period. From each run, a realization of the fraction of cooperators is obtained by averaging the last timesteps (i.e. once the dynamics is almost stabilized).

Undirected graph: An undirected graph results from bilateral linking choices.

Unilateral linking choice: Unilateral linking choice means that no consent is necessary to establish a link, i.e. the existence of a link from i to j does not imply the existence of a link from j to i.

Appendix B. Proofs

Proof of Proposition 1. \(i\) If \(\eta_i < \eta\) potential partners for i have to be link constrained, i.e. \(\forall j \in N_i^{1+Z}\setminus N_i^1 : \eta_j = \eta\). Else i and j would form a link. \(ii\) States with only monomorphic components, where \(i\) holds, are absorbing, as no agent has “possibilities to imitate” because card \(N_i(-z_i) = 0\). \(iii\) We first consider incentives to change links and show that in any absorbing state there cannot exist two cooperators i and i’ separated by a path of defectors of any length. If \(i’ \in N_i^{1+Z}, i\) and \(i’\) will form a link. If not, cooperator i will form a link with defector j at distance of at most \(1+Z\). This link \(ij\) will be formed because the cooperator is always willing to sever a link with a defector. On the other hand defector \(j’\) (connected to \(i\) through...
some path of defectors) is willing to form a link with i, whenever \( N_i^c \cap N_j^c \) contains only defectors, as in this case \( \Pi_{\min}^{i-1}(N_i^c) = d < T^{i-1}(N_i^c \cap N_j^c) \). Repeating this argument it can be seen that the distance between i and i' gets shorter and shorter until finally \( i' \in N_i^{+2} \). But then i and i' will link and all mixed links will eventually be cut. It follows analogously that each defector j must lie at a distance of at most \( l + Z \) from cooperators.

Next we show that such states are indeed absorbing under the conditions in Proposition 1(iii). A sufficient condition is that defectors form a clique (i.e. are all linked with each other). If either \( l \leq 2 \) or \( Z = 1 \) this is also a necessary condition. We start with linking deviations. Assume that either \( l = 1 \) or \( Z = 1 \) and that there is only one cooperator i linked to some of a set of defectors. Any defector at a distance of at most \( l + Z \) has incentives to link to cooperators. If \( l = 1 \), defectors observe only defectors interacting with cooperators. But then \( \Pi_{\min}^{i-1}(N_i^c) = d < T^{i-1}(N_i^c \cap N_j^c) \). If \( l > 1 \), any defector may observe in addition cooperators other than i, but since \( Z = 1 \) there cooperators interact only with cooperators and again \( \Pi_{\min}^{i-1}(N_i^c) = d < T^{i-1}(N_i^c \cap N_j^c) \). Thus such links might be formed. This rewiring can be part of a recurrent set if and only if \( N_i^{l+2} \) remains unchanged. It follows that the set of defectors must form a clique. Now assume \( l > 1 \) and \( Z > 1 \). Again cooperators i has incentives to sever any of her mixed links. The incentives of i's potential partners depend on how many cooperators interact with the defectors they observe. To characterize all structures in this case is impossible without further assumptions.

Finally consider agents' incentives to change actions. Assume that x defectors form a clique and that there is only one cooperator i linked with them.\[^{41}\] We show that there exists a threshold for the interaction radius, \( \hat{Z}(l) \) such that if \( Z < \hat{Z}(l) \) there always exists payoffs for which such an action profile is absorbing. To simplify the exposition we normalize \( c = 1 \) and \( b = 0 \). It should be clear that if i does not want to change action, then no other cooperators has incentives to do so. For cooperators i and any defector j in the clique,

\[
\Pi_i(N_i^c(D)) = \Pi_j(N_i^c(D)) = \Pi_j = \frac{n_j^2 - (x-1) + (x-1)d}{n_j^2}.
\]

Action choices are absorbing if and only if \( \Pi_i(N_i^c(C)) \geq \Pi_j \geq \Pi_j(N_i^c(C)) \). Now we show that for each i, there exists \( \hat{Z}(l) \) s.t. if \( Z < \hat{Z}(l) \), it is always possible to find payoff parameters such that the previous inequality is true. First of all note that \( \Pi_j = (n_j^2 - (x-1) + (x-1)d)/n_j^2 \) is monotonously increasing in both Z and d. The sample payoffs \( \Pi_i(N_i^c(C)) \) and \( \Pi_j(N_i^c(C)) \) are increasing in \( a \). If \( Z < \hat{Z}(l) \) an increase in Z has two effects. On the one hand each cooperators in the set \( N_i^c \) interacts with more cooperators increasing the sample payoff. But on the other hand, more cooperators interact with defectors lowering the sample payoffs. The net effect depends on the precise structure of the component. Consider first the case where Z is small, in particular where \( Z = 1 \). Then \( \Pi_j = (n_j^2 - (x-1) + (x-1)d)/n_j^2 \leq 1/2 \). On the other hand, for any \( a > 1/2 \), \( \Pi_i(N_i^c(C)) = ((\eta - x)/\eta)a + \phi_j(l)a)/(\phi_j(l) + 1) \) and \( \Pi_j(N_i^c(C)) = ((\eta - x)/\eta)a + \phi_j(l)a)/(\phi_j(l) + 1) \), where \( \phi_j(l) > \phi_j(l) \) are, respectively, the number of cooperators i contained in \( N_i^c \) and \( N_j^c \). Then whenever \( a > (1/\eta)\Pi_j/(\eta - x) \),

\[
\Pi_i(N_i^c(C)) > \Pi_j(N_j^c(C)) > \Pi_j(d = 0) \approx \frac{1}{x}.
\]

On the other hand for Z very large,

\[
\Pi_j(d = 0) \rightarrow 1 > \Pi_i(N_i^c(C)) \approx \Pi_j(N_j^c(C)) \rightarrow a.
\]

Consequently there exists a threshold value \( \hat{Z}(l) \), such that if \( Z < \hat{Z} \) there always exists payoff parameters for which there are no incentives to imitate actions.\[^{42}\]

\[^{41}\] Of course \( x \geq 2 \) has to hold.

\[^{42}\] Note also that \( \Pi_j(d = a) = ((x-1)a + 1)/x > a \) \( \Pi_j(N_j^c(C)) \) (no intersection).

\[^{43}\] See also Young (1993, 1998).

\[^{44}\] See also Young (1993, 1998).
defined as the sum of minimal mutations necessary to induce a (possibly indirect) transition to \( s \) from any alternative state \( s' \in \Omega \), i.e., \( \psi(s) = \sum_{s \in \Omega} \psi(X(s,s')) \).

**Result** (Young, 1993). State \( s^* \) is stochastically stable if it has minimal stochastic potential, i.e., if \( s^* \in \arg\min_{s \in \Omega} \psi(s) \).

**Lemma 1** (Topology). If \( 1+Z > 2 \), all polymorphic stochastically stable states will consist of at most two disconnected components and all monomorphic stochastically stable states will be connected.

**Proof.** Let \( G^0 \) denote the set of graphs consisting of at most two disconnected components. Let \( G^1 \) be the set of graphs one tremble away from some network in \( G^0 \). Define \( G^2 \) to be graphs not in \( G^0 \cup G^1 \) that are one tremble away from \( G^1 \). For \( \tau > 2 \) let \( G^\tau \) denote graphs not in \( G^\tau \) for any \( \tau < \tau \), that are one tremble from \( G^\tau \). Note that these exhaust all graphs that could be part of absorbing sets. Consider an absorbing state graph \( G^\tau \), \( \tau > 0 \). Transitions from \( G^\tau \) to some \( G^\tau+1 \) can occur after just one tremble, as it is always possible that two players \( i \) and \( h \), with \( \xi(i) \neq \xi(h) \) and \( z_i = z_h \) form a link by mistake. This implies that for any \( s' \in G^\tau \), there exists \( s' \in G^\tau+1 \) s.t. \( \psi(s') \leq \psi(s) \). (Starting from an \( s' \)-tree one can always redirect an arrow from \( s' \) to \( s \), which is one tremble away). Thus to complete the proof we show (i) that the stochastic potential of states with a graph in \( G^\tau \) is smaller than that of states with a graph in \( G^\tau+1 \) and (ii) that the stochastic potential of connected monomorphic states is smaller than that of monomorphic states where graphs consist of two disconnected components. Start with the absorbing state \( s \) with \( G \in G^1 \) and find a state \( s' \) with graph \( G \in G^0 \). We know that \( X(s,s') = 1 \) and \( X(s',s) = 1 \). We will now see in which cases strict inequality obtains.

Consider first the transition through which \( s' \) is reached from \( s \). For this transition a link of \( s' \) is formed by mistake between \( i \) and \( h \) s.t. \( \xi(i) \neq \xi(h) \) and \( z_i = z_h \). If \( i \) and \( h \) have neighbors, say \( j \) and \( k \), that are not linking constrained, then, whenever \( l+i+Z > 2 \), (at least) the link \( jk \) will be formed before an absorbing state is reached. But then at least two trembles are needed for the transition \( s' \rightarrow s \) and consequently \( X(s,s') > 1 \). Note that such two states \( s' \) and \( s \) can always be found. What happens if for two states \( s \) with \( G^0 \) and \( s' \) with graph \( G^1 \) we have that \( X(s,s') = 1 ? First note that for any \( s' \) a state \( s' \) with \( G^1 \) can be found such that (a) \( X(s',s) > 1 \) and (b) \( s' \) can be reached from \( s \) via a series of “one-trembles.” But then we have that \( \psi(s') \leq \psi(s) \). Focus thus on states \( s' \) with graph \( G^0 \) where \( X(s',s) > 1 \) and \( X(s,s) = 1 \) for some state \( s \) with \( G^1 \). Then starting from a minimal \( s' \)-tree, add an arrow \( s' \rightarrow s \). Consider the old path \( s' \rightarrow s \) and take the first \( s' \) on that path (this could be \( s' \)) such that the arrow pointing away from \( s' \) involves at least two trembles. Cut this arrow. Note that such a state \( s' \) must exist because at some point (at least) two links have to be severed to separate the component of players. In effect, \( s' \) must have a graph in \( G^2 \) and to separate the component at least two trembles will be needed: any two agents \( i \) and \( h \) such that in \( s' : \xi(i) \neq \xi(h) \) who cut a link starting from \( s' \) will be in each other’s search radius and thus for \( s' \) to be absorbing either have to form a link (but then \( s' \) is \( s' \)) or either of them has to form a link with another agent). Then starting from a \( s' \)-tree we have created an \( s' \)-tree, by cutting an arrow with a “cost” exceeding two and adding an arrow with a cost of one. Consequently we have shown that for any \( s' \) with \( G^0 \) there exists a state \( s'' \) with graph \( G'' = G^0 \) s.t. \( \psi(s'') < \psi(s) \). The argument can be repeated starting from a monomorphic state \( s' \) with two disconnected components. This completes the proof. \( \square \)

In the following we will denote by \( \omega^D_{\rho} \) the set of absorbing states where all agents play action \( z \) and where the network consists of \( \rho \) disconnected components. Denote by \( \bigcup_{\rho \in [1..n]} \omega^D_{\rho} = \omega^D \). Analogously \( \omega^{C\rho} \) is the set of all polymorphic absorbing states with \( \rho \) components.

**Lemma 2** (Instability of full cooperation). States \( s \in \omega^C \), where all agents cooperate, are not stochastically stable.

**Proof.** It follows from Lemma 1 that if stochastically stable states that involve cooperation exist at least one of them has to be connected, i.e., it has to be contained in the set \( \omega^C \). We will now show that for any \( s \in \omega^C \) there exists an alternative state in \( \omega^{C\rho} \) that has strictly less stochastic potential. For any \( s \in \omega^D \) consider the state \( s'' \in \omega^{C\rho} \) reached via one tremble from \( s \) in the following way. Assume one player \( i \) trembles and switches to action \( D \). Then for all agents \( j \in N \) the average payoff of action \( D \) will exceed that of action \( C \). Assume \( z \) agents selected from that set switch to action \( D \) and that the subgraph containing these agents is cut off (through rewiring of cooperating neighbors who prefer being linked to a cooperators) only after \( K_D > n \) agents in total (including the mutant) have switched to \( D \). Note that irrespective of the payoff parameters and of \( i \) and \( Z \) this is always possible. State \( s'' \) contains thus two disconnected components, one consisting of \( K_D > n \) defectors and one of \( n-K_D \) cooperators. The reverse transition \( (s'' \rightarrow s) \) will need at least 2 trembles, as one tremble tremble has to occur to merge the two components and in addition at least one of the defectors has to tremble to switch to cooperation. (Note again that any single (non-isolated) defector will have a higher per interaction payoff than cooperators). Next take a minimal \( s'' \)-tree and add the arrow \( s'' \rightarrow s' \) at a cost of \( X(s,s') = 1 \). Then consider the path \( s'' \rightarrow s' \). If there is no other state on this path, cut the arrow \( s'' \rightarrow s' \). This yields an \( s'' \)-tree with \( \psi(s'') < \psi(s) \). If there is a state \( s'' \in \omega^D \) on this path, then we know that \( X(s,s') > 1 \) (because a single defector in a component of defectors will never be imitated). We can cut the arrow \( s'' \rightarrow s' \) and have constructed again an \( s'' \)-tree with \( \psi(s'') < \psi(s) \). If \( s'' \in \omega^{C\rho} \) then we know that \( X(s,s') > 2 \) by the same argument as above. Cutting the arrow \( s'' \rightarrow s' \) leaves us with a \( s'' \)-tree with \( \psi(s'') < \psi(s) \). This completes the proof. \( \square \)

**Distance between graphs:** Before stating the next proof let us introduce the following metric. Define \( \gamma(G,G') = \sum_{l} |(l_l-l'_l)/2 \) to be the distance between the graphs \( G \) and \( G' \) associated with states \( s \) and \( s' \), respectively. The distance \( \gamma(G,G') \)

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Note that if starting from \( s \) the component is separated at least two trembles are needed and thus \( s'' \) is \( s'' \).

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between two graphs simply measures the number of links that differ between the two graphs. Furthermore denote by \( c_{ij}(t) \) the share of agents \( j,k \in N^C_t \) at time \( t \) that are \( Z \)-th order neighbors themselves. \( c_{ij}(t) \) is a measure of local clustering in \( f \)’s interaction neighborhood.

**Lemma 3** (Polymorphic states—II). \( \forall s \in \omega^P_{ij}, 3a(s) \in (d,c) \) s.t. whenever \( a > a(s) : 3\epsilon \in \omega^P_{ij}, \rho \leq 2 \) with \( X(s,s') < X(s,s) \).

**Proof.** (i) Starting from a state \( s \in \omega^P_{ij} \) we construct a state \( s' \in \omega^P_{ij} \) as follows. Assume that \( |\epsilon C| \) agents (where \( \epsilon C \in \mathbb{R} \)) tremble and switch to action \( C \) at time \( t \). We want to consider the action choice of a defector \( k \) linked with a cooperator \( i \). Assume that all other cooperators are (1st-2nd-... \( Z \)-th order) neighbors of \( i \), i.e. are all interacting with \( i \). The sample payoff of cooperation that agent \( k \) observes is given by \( \Pi(N^C_k(t)) = b + (a-b)h(\epsilon C, n^2, c_{ij}(t)) \) where \( g(\cdot) \) is an increasing function of clustering and of \( \epsilon C \). On the other hand the sample payoff of defection that agent \( k \) observes is given by \( \Pi(N^D_k(t)) = d + (c-d)g(\epsilon C, n^2, c_{ij}(t)) \) where \( g(\cdot) \) is a decreasing function of clustering and of \( \epsilon C \). Denote the value of \( \epsilon C \) that solves \( \Pi(N^C_k(t)) = \Pi(N^D_k(t)) \) by \( \epsilon C^* \). This value is in general a complicated expression but note that \( (\epsilon C^*/\epsilon C < 0 \). Now whenever agent \( k \) has incentives to switch to cooperation (i.e. whenever \( \Pi(N^C_k(t)) > \Pi(N^D_k(t)) \)) then \( \epsilon C \geq n^2 - 1 - \epsilon C \) agents can be infected through the ensuing operation of the unperturbed action dynamics alone.

Through the operation of the unperturbed linking dynamics, all cooperators will sever their remaining links with defectors and form links among each other. (Note that this is possible because \( \epsilon C + \epsilon C \geq 1 \) so these agents can always at least form the complete component. Furthermore they have incentives to do so, as \( \Pi(N^C_k(t)) = \Pi(N^D_k(t)) \) for any pair of cooperating agents \( j,h \). Note also that by construction all these agents are in each other’s search set.)

(ii) Consider the reverse transition from \( s' \in \omega^P_{ij} \) to \( s \in \omega^P_{ij} \). Essentially such a transition can occur in two ways. Either the cooperative component \( \chi^C(s) \) is first infected by defection and then the graph is rewired to obtain state \( s \). (In this case the transition is indirect, i.e. passes through other absorbing states among which at least one is in \( \omega^P_{ij} \).) Or first a sufficient number of linking trembles has to occur s.t. the ensuing operation of the unperturbed dynamics permits infecting all agents with defection while rewiring the graph. (In this case the transition is direct.)

Consider the first type of transition. For this transition \( \epsilon C^* \) action trembles are needed to infect the cooperative component and then \( \epsilon C^* \) linking trembles are needed to rewire the graph. Note now that while \( X(s,s') = \epsilon C^* + \epsilon C^* \) (\( \epsilon C^* \) \& \( \epsilon C \) ) is strictly increasing with the payoff parameter \( a \in (d,c) \), \( X(s,s') \) is decreasing in \( a \). Consequently there exists \( a(s) \) s.t. \( X(s,s') > X(s,s') \) when \( a > a(s) \). Now consider the second type of transition. First note that a cooperating agent \( i \in \chi^C(s) \) linked to a defector \( j \in \chi^D(s) \) (after a linking tremble) has incentives to switch to defection if and only if

\[
a < \frac{n^2(\chi^C)}{\chi^C} \frac{1}{z^2} b = \frac{1}{z^2} \frac{a - \epsilon C^* b}{1 - \epsilon C^*},
\]

where the factor \( n^2(\chi^C) \) gives the ratio of cooperators and defectors in the set \( N^C_i \) and \( z^2(\chi^C) \) is the share of defectors these defectors (cooperators) interact with on average. Note also that whenever (B.1) fails no links will be formed between neighbors \( h \) of \( i \) and neighbors \( k \) of \( j \), unless \( h \) has a neighbor who is playing defection. (If \( h \) does not have a defector neighbor, then \( \Pi(N^C_i \bar{N}^C_i) > \Pi(N^C_i \bar{N}^C_i) \) if \( j \) belongs to \( N^C_i \) \& \( N^C_i \) or \( i \) belongs to \( N^C_i \) \& \( N^C_i \). But if \( j \) belongs to \( N^C_i \) \& \( N^C_i \), i.e. if \( N^C_i \) \& \( N^C_i \) \& \( \chi^C(s) \) = \( \emptyset \) then a failure of (B.1) implies \( \Pi(N^C_i \bar{N}^C_i) > \Pi(N^C_i \bar{N}^C_i) \).) The number of trembles needed to induce such a transition is thus strictly increasing with the payoff parameter \( a \). Consequently there exists a threshold value \( \bar{a}(s) \) such that whenever \( a > \bar{a}(s) \), \( X(s,s') > X(s,s') \). Thus whenever \( a > a(s) = \max(\bar{a}(s), \bar{a}(s)) \) we have that \( X(s,s') < X(s,s) \). This completes the proof. □

**Lemma 4** (Polymorphic states—II). If \( I + Z > 2 \) states in \( s \in \omega^P_{ij} \) are not stochastically stable.

**Proof.** Starting from any polymorphic absorbing state \( s \in \omega^P_{ij} \) with \( \rho 


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\(^{45}\) This metric has been used previously by Goyal and Vega-Redondo (2005).
on this path will be contained in \( \omega^F \) or \( \omega^D \) (with the exception of the state \( s' \)). (a) If \( (s', \ldots, s) = (s', s) \) i.e. if the transition from \( s' \) to \( s \) is direct we can infer immediately that \( \psi(s') < \psi(s) \). (b) Next assume that there exists a state \( s' \in (s, \ldots, s) \) with \( s' \in \omega^D \). Note that \( X(s', s) > X(s, s) \) always holds under the assumption that \( a > a' \), as can be read from the proof of Lemma 3. But if \( X(s', s) > X(s, s) \) we can find an \( s' \)-tree with \( \psi(s') < \psi(s) \) simply adding the arrow \( s' \rightarrow s \) and deleting the arrow \( s' \rightarrow s \). Thus \( s \) cannot be stochastically stable. On the other hand it follows from Lemma 1 that states in \( \omega^D \) where \( \rho > 1 \) cannot be stochastically stable either. (c) Furthermore it follows from the proof of Lemma 3 that whenever the path \( (s', \ldots, s) \) in a minimal \( s \)-tree contains a state \( s' \in \omega^F \), it also contains a state \( s' \in \omega^D \). But we have already seen that in this case \( s \) is not stochastically stable. Consequently all stochastically stable states are contained in \( \omega^D \) where \( \rho < 2 \). The remaining results follow directly from Lemmas 1 and 4.

References


